Solving Robust Binary Optimization Problem with Budget Uncertainty

Christina Büsing, Timo Gersing, Arie Koster

Mixed Integer Programming Workshop, 2025

Dicrete Optimization and MIPs

Mixed Integer Program

$$
\begin{aligned}
\min c^\top x& \\
Ax &\geq b \\
x &\in \{0,1\}
\end{aligned}
$$

Dicrete Optimization and MIPs

Mixed Integer Program

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\min\ & c^\top x \\
& Ax \geq b \\
& x \in \{0,1\}
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Mixed Integer Program

$$\begin{aligned} \min\ & c^\top x \\ & Ax \geq b \\ & x \in \{0,1\} \end{aligned}$$

Historical Data/Measurements

506160	16AUG2011:1	I1-SM/ICD	SM-2KAMME	24AUG2012:1	24AUG2012:1	1,5824E+12	I1	I1LST	5	Im Kalender	16AUG2011:1	16AUG2011
511769	17AUG2011:1	I1-UNTERS	WDH	16AUG2012:0	16AUG2012:0	2,325E+11	I1	I1ECHO	5	Im Kalender	17AUG2011:1	17AUG2011
564409	30AUG2011:1	I1-SM/ICD	SM-1KAMME	28AUG2012:1	28AUG2012:1	3,7552E+12	I1	I1LST	5	Im Kalender	30AUG2011:1	30AUG2011
569745	31AUG2011:1	I1-SM/ICD	SM-1KAMME	29AUG2012:1	29AUG2012:1	7,2358E+12	I1	I1LST	5	Im Kalender	31AUG2011:1	31AUG2011
569745	31AUG2011:1	I1-SM/ICD	SM-1KAMME	29AUG2012:1	29AUG2012:1	7,2358E+12	I1	I1LST	5	Im Kalender	31AUG2011:1	31AUG2011
569745	31AUG2011:1	I1-SM/ICD	SM-1KAMME	29AUG2012:1	29AUG2012:1	7,2358E+12	I1	I1LST	5	Im Kalender	31AUG2011:1	31AUG2011
644371	20SEP2011:1	I1-UNTERS	WDH	17SEP2012:0	17SEP2012:0	3,423E+12	I1	IM1K	5	Stornierungs	19SEP2011:0	20SEP2011:
644372	20SEP2011:1	I1-UNTERS	GKP+BGA+DI	17SEP2012:0	17SEP2012:0	3,423E+12	I1	IM1K	5	Stornierungs	19SEP2011:0	20SEP2011:
648823	29SEP2011:1	I1-SM/ICD	SM-2KAMME	26SEP2012:1	26SEP2012:1	2,5708E+12	I1	I1LST	5	Im Kalender	19SEP2011:1	29SEP2011:
711316	04OCT2011:1	I1-SM/ICD	SM-2KAMME	25SEP2012:1	25SEP2012:1	6,3114E+11	I1	I1LST	5	Im Kalender	04OCT2011:1	04OCT2011
743860	11OCT2011:1	I1-SM/ICD	SM-2KAMME	06SEP2012:1	06SEP2012:1	3,9465E+12	I1	I1LST	5	Im Kalender	11OCT2011:1	11OCT2011
760872	14OCT2011:1	I1-SM/ICD	SM-1KAMME	11JUL2012:1	11JUL2012:1	2,9754E+11	I1	I1LST	5	Im Kalender	14OCT2011:1	14OCT2011
533627	12DEC2011:0	I1-SM/ICD	ICD-BIVENT	10AUG2012:1	10AUG2012:1	1,4495E+10	I1	I1LST	5	Stornierungs	23AUG2011:1	12DEC2011
034844	16DEC2011:1	I1-SM/ICD	ICD-2KAMME	15JUN2012:1	15JUN2012:1	1,264E+12	I1	I1LST	5	Im Kalender	16DEC2011:1	16DEC2011
845266	03NOV2011:1	I1-SM/ICD	SM-2KAMME	27JUL2012:0	27JUL2012:0	1,2934E+12	I1	I1LST	5	Im Kalender	03NOV2011:1	03NOV201
817375	27OCT2011:1	MED1	HOLTER3TAG	16JUL2012:1	16JUL2012:1	3,5175E+12	I1	I1PO	10	Im Kalender	27OCT2011:1	27OCT2011
817375	27OCT2011:1	MED1	HOLTER3TAG	16JUL2012:1	16JUL2012:1	3,5175E+12	I1	I1PO	10	Status über	27OCT2011:1	27OCT2011
147160	06FEB2012:1	I1-SM/ICD	ICD-1KAMME	13JUL2012:1	13JUL2012:1	5,6679E+12	I1	I1ECHO	5	Res. ROM/I1	13JAN2012:1	06FEB2012:
112889	06JAN2012:1	I1-ECHOKAR	TTE	23MAY2012:0	23MAY2012:0	3,962E+11	I1	I1ECHO	10	Im Kalender	06JAN2012:1	06JAN2012
112889	06JAN2012:1	I1-ECHOKAR	TTE	23MAY2012:0	23MAY2012:0	3,962E+11	I1	I1ECHO	10	Status über	06JAN2012:1	06JAN2012
254641	06FEB2012:1	I1-SM/ICD	ICD-2KAMME	02AUG2012:0	02AUG2012:0	3,0771E+12	I1	I1LST	5	Im Kalender	06FEB2012:1	06FEB2012:
254641	06FEB2012:1	I1-SM/ICD	ICD-2KAMME	02AUG2012:0	02AUG2012:0	3,0771E+12	I1	I1LST	5	Stornierungs	06FEB2012:1	06FEB2012:

PILOT4 from NETLIB library

- Constraint 372

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\geq\ & b \equiv 23.387405
\end{aligned}
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- Optimal "classical" solution

$$\begin{aligned} x^*_{826} &= 255.6112787181108 & x^*_{827} &= 6240.488912232100 \\ x^*_{828} &= 3624.613324098961 & x^*_{829} &= 18.2020506528325 9 \\ x^*_{849} &= 174397.0389573037 & x^*_{870} &= 14250.00176680900 \\ x^*_{871} &= 25910.00731692178 & x^*_{880} &= 104958.3199274139 \end{aligned}$$

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- Can the coefficients be known with such a high accuracy?

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- Can the coefficients be known with such a high accuracy?
- Assume 0.1%-accurate approximation

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- Worst violation: 450%

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- Random perturbation: $(1 + \xi_j)a_j$

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- Relative violation:

$$V = \frac{b - \tilde{a}^T x^*}{b} \times 100\%$$

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- Prob$\{V > 0\} = 0.5$
- Prob$\{V > 150\%\} = 0.18$
- Mean$(V) = 125\%$

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- Robust optimization: robust solutions remain (almost) always feasible

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- Prob$\{V > 0\} = 0.5$
- Prob$\{V > 150\%\} = 0.18$
- Mean$(V) = 125\%$

- Robust optimization: robust solutions remain (almost) always feasible
- Usually still very good objective value

Definition (Robust Optimization with interval cost uncertainties)

Given a set of feasible solution $\mathcal{X} \subseteq \{0,1\}^n$. Let $\mathcal{U}$ be an interval uncertainty set defining different costs $\mathbf{c} \in [\underline{c}, \overline{c}]^n$. A *robust* optimial solution $\mathbf{x}^* \in \mathcal{X}$ is a feasible solution minimizing the worst case costs, i.e., solve

$$\min_{\mathbf{x} \in \mathcal{X}} \left\{ \max_{\mathbf{c} \in \mathcal{U}} \sum_{i=1}^{n} c_i x_i \right\}$$

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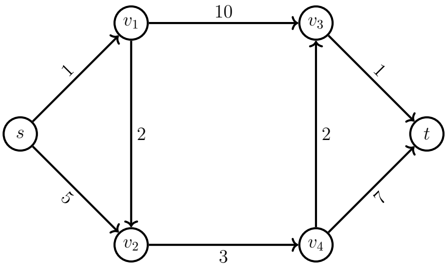
Shortest Path Problem



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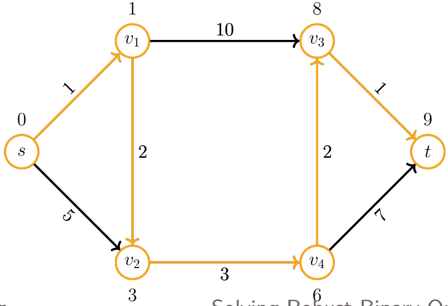

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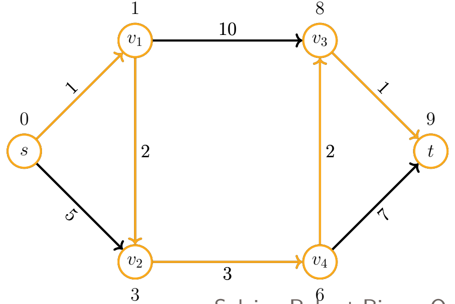
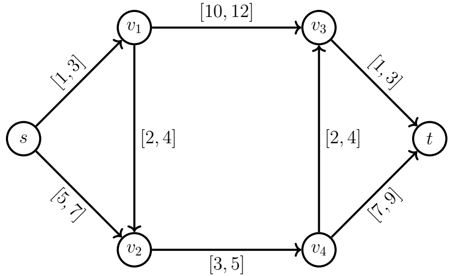
Shortest Path Problem





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Given a set of feasible solution $\mathcal{X} \subseteq \{0,1\}^n$. Let $\mathcal{U}$ be an interval uncertainty set defining different costs $\mathbf{c} \in [\underline{c}, \overline{c}]^n$. A *robust* optimial solution $\mathbf{x}^* \in \mathcal{X}$ is a feasible solution minimizing the worst case costs, i.e., solve

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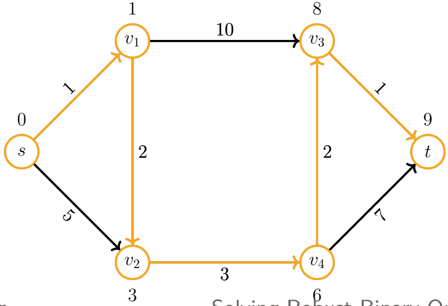

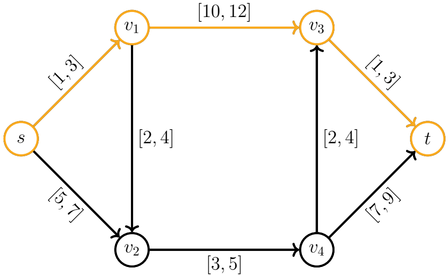

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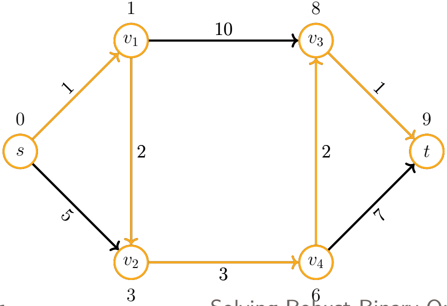

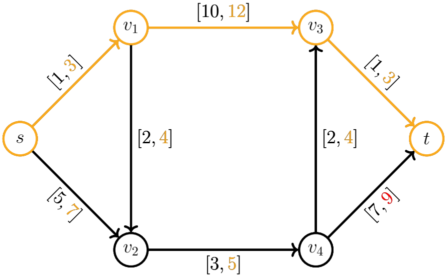

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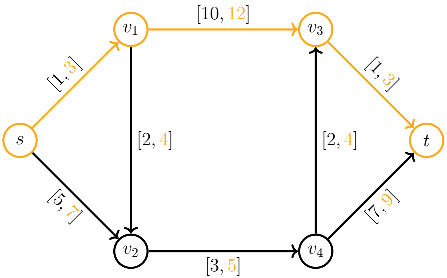

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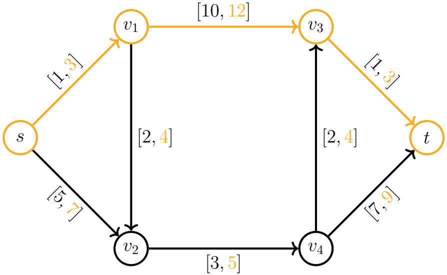

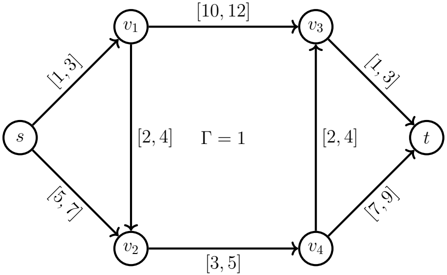

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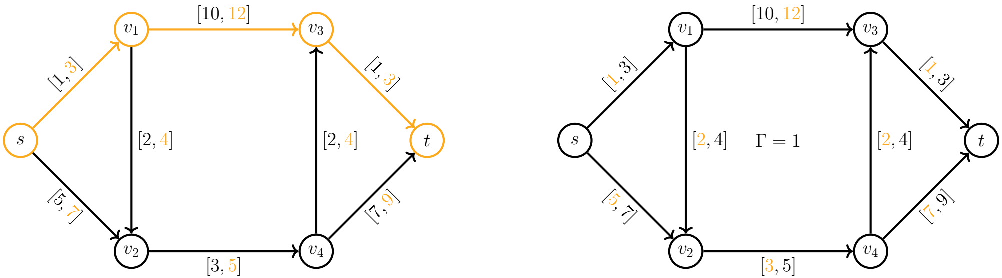

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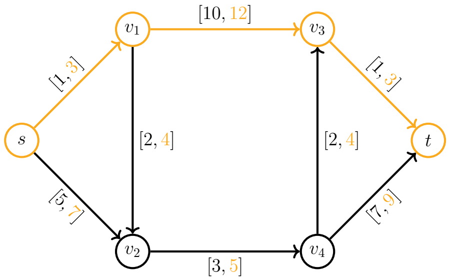

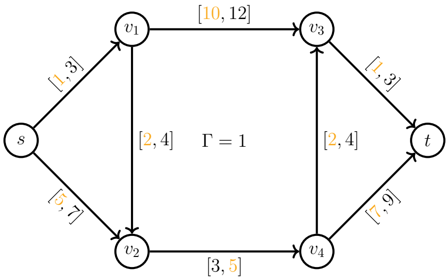

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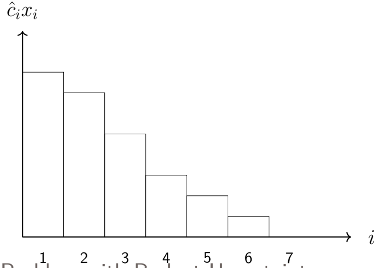

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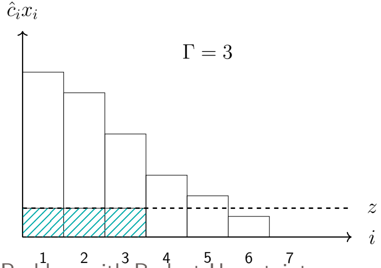

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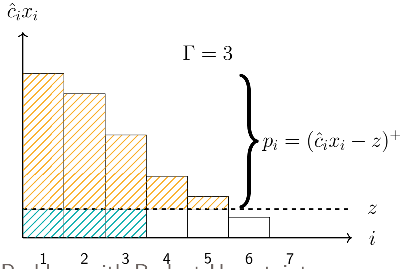

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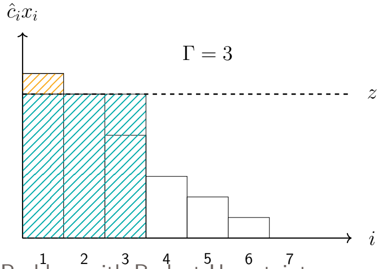

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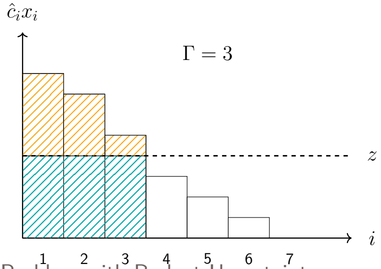

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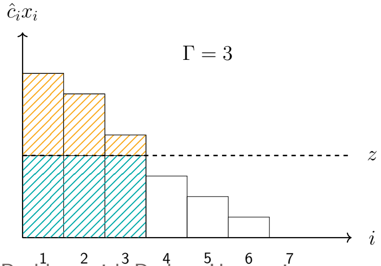

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Practical Performance

- compact formulation, no use of big M
- let's solve some problems!
- robust knapsack:

```
                Nodes                                                Cuts/
    Node   Left      Objective  IInf  Best Integer     Best Bound     ItCnt      Gap

 23720047 15197259    11719,6862    20    11606,0000    12288,1353 55057467    5,88%
 23770901 15226459    11828,1686    23    11606,0000    12287,7700 55180967    5,87%
 23821565 15255529    12011,4041    24    11606,0000    12287,4030 55303331    5,87%
 23871269 15283781    11783,7154    22    11606,0000    12287,0473 55424214    5,87%
Elapsed time = 3434,41 sec. 5979817,73 ticks, tree = 3353,00 MB, solutions = 12
Nodefile size = 1305,88 MB 727,49 MB after compression
 23922166 15312936    12191,2914    28    11606,0000    12286,6809 55547450    5,86%
 23972875 15342043    12200,0047    28    11606,0000    12286,3216 55668913    5,86%
 24023053 15370543    11736,4889    21    11606,0000    12285,9418 55790174    5,86%
 24073375 15399019    12100,5997    25    11606,0000    12285,5821 55912495    5,86%
 24124016 15427933    12185,6295    27    11606,0000    12285,2237 56034120    5,85%
 24174475 15456520    12076,8979    25    11606,0000    12284,8592 56156283    5,85%
 24223910 15484613    11936,4399    23    11606,0000    12284,4984 56276599    5,85%
 24273972 15512958    11751,6692    24    11606,0000    12284,1408 56398552    5,84%
 24324148 15541333    11929,2290    23    11606,0000    12283,7841 56521083    5,84%
 24374451 15569768    12255,0014    27    11606,0000    12283,4225 56644319    5,84%
Elapsed time = 3528,01 sec. 6132528,45 ticks, tree = 3408,61 MB, solutions = 12
Nodefile size = 1360,88 MB 755,91 MB after compression
 24424125 15598043    11993,2112    23    11606,0000    12283,0698 56764602    5,83%
 24475158 15626928    12240,6841    27    11606,0000    12282,7013 56887548    5,83%
 24526113 15655874    12232,6254    27    11606,0000    12282,3386 57011130    5,83%
 24576245 15684253    12236,5555    30    11606,0000    12281,9808 57133348    5,82%
 24625778 15712276        cutoff          11606,0000    12281,6332 57253146    5,82%
 24676376 15740977    11992,7508    26    11606,0000    12281,2759 57374978    5,82%
 24726652 15769305    12240,2652    28    11606,0000    12280,9179 57496901    5,82%
 24777038 15797704    11615,6468    22    11606,0000    12280,5627 57618421    5,81%
 24827584 15826342    12045,1031    24    11606,0000    12280,2089 57740201    5,81%
 24877740 15854780        cutoff          11606,0000    12279,8571 57860645    5,81%
Elapsed time = 3623,59 sec. 6285119,25 ticks, tree = 3464,00 MB, solutions = 12
Nodefile size = 1416,87 MB 784,63 MB after compression
```

- compact formulation, no use of big M
- let's solve some problems!
- robust knapsack:

```
        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

 23720047 15197259    11719,6862    20    11606,0000    12288,1353 55057467    5,88%
 23770901 15226459    11828,1686    23    11606,0000    12287,7700 55180967    5,87%
 23821565 15255529    12011,4041    24    11606,0000    12287,4030 55303331    5,87%
 23871269 15283781    11783,7154    22    11606,0000    12287,0473 55424214    5,87%
Elapsed time = 3434,41 sec. 5979817,73 ticks, tree = 3353,00 MB, solutions = 12
Nodefile size = 1305,88 MB 727,49 MB after compression
 23922166 15312936    12191,2914    28    11606,0000    12286,6809 55547450    5,86%
 23972875 15342043    12200,0047    28    11606,0000    12286,3216 55668913    5,86%
 24023053 15370543    11736,4889    21    11606,0000    12285,9418 55790174    5,86%
 24073375 15399019    12100,5997    25    11606,0000    12285,5821 55912495    5,86%
 24124016 15427933    12185,6295    27    11606,0000    12285,2237 56034120    5,85%
 24174475 15456520    12076,8979    25    11606,0000    12284,8592 56156283    5,85%
 24223910 15484613    11936,4399    23    11606,0000    12284,4984 56276599    5,85%
 24273972 15512958    11751,6692    24    11606,0000    12284,1408 56398552    5,84%
 24324148 15541333    11929,2290    23    11606,0000    12283,7841 56521083    5,84%
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 24877740 15854780        cutoff          11606,0000    12279,8571 57860645    5,81%
Elapsed time = 3623,59 sec. 6285119,25 ticks, tree = 3464,00 MB, solutions = 12
Nodefile size = 1416,87 MB 784,63 MB after compression
```

Practical Performance

- compact formulation, no use of big M
- let's solve some problems!
- robust knapsack: 50 items can already be intractable

```
         Nodes                                              Cuts/
   Node  Left      Objective  IInf  Best Integer     Best Bound     ItCnt     Gap

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 23770901 15226459    11828,1686    23    11606,0000    12287,7700 55180967    5,87%
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Elapsed time = 3623,59 sec. 6285119,25 ticks, tree = 3464,00 MB, solutions = 12
Nodefile size = 1416,87 MB 784,63 MB after compression
```

$$
\begin{aligned}
\min \quad & \Gamma z + \sum_{i \in N} p_i + \sum_{i \in N} c_i x_i \\
\text{s.t.} \quad & Ax \geq b \\
& z + p_i \geq \hat{c}_i x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n_{\geq 0}, z \geq 0, x \in \{0,1\}^n
\end{aligned}
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Strong Formulations

- Atamtürk:
  four strong versions

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$$

Strong Formulations

- Atamtürk:
  four strong versions

Discretize z

- Bertsimas & Sim:
  $n+1$-subproblems
- Hansknecht et. al:
  Divide and Conquer

- Atamtürk Formulations: If the nominal formulation is α-tight then the strongest formulation is also α-tight for the robust problem [At2006]

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- Remedy: multiply z with x_i to strengthen the constraint:

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\begin{aligned}
\min \quad & \sum_{i \in N} c_i x_i + \Gamma z + \sum_{i \in N} p_i \\
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Theorem

The above bilinear formulation is stronger than any polyhedral formulation.

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Theorem

The above bilinear formulation is stronger than any polyhedral formulation.

- The bilinear formulation is impractical but the starting point for two new approaches

- For fixed $z = z'$ it holds
  $p_i = (\hat{c}_i x_i - z')^+ = (\hat{c}_i - z')^+ x_i$

$$
\begin{aligned}
\min \quad & \Gamma z + \sum_{i \in N} p_i + \sum_{i \in N} c_i x_i \\
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- Fixing z yields a nominal problem

$$\begin{aligned} \min \quad & \Gamma z' + \sum_{i \in N} (c_i + (\hat{c}_i - z')^+) x_i \\ \text{s.t.} \quad & Ax \geq b \\ & x \in \{0,1\}^n \end{aligned}$$

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- The Γ largest value $\hat{c}_i x_i$ is an optimal choice for z

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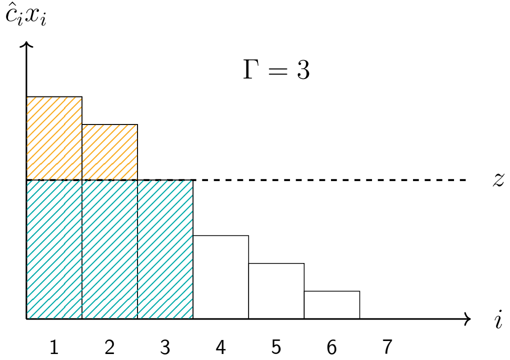

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- The Γ largest value $\hat{c}_i x_i$ is an optimal choice for z
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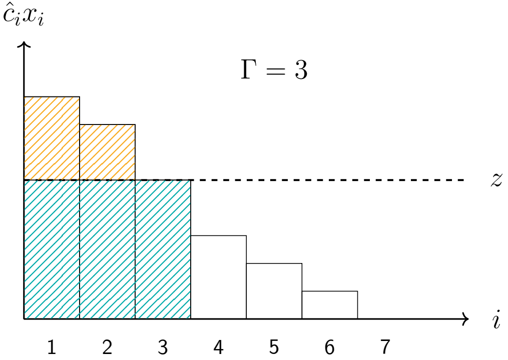

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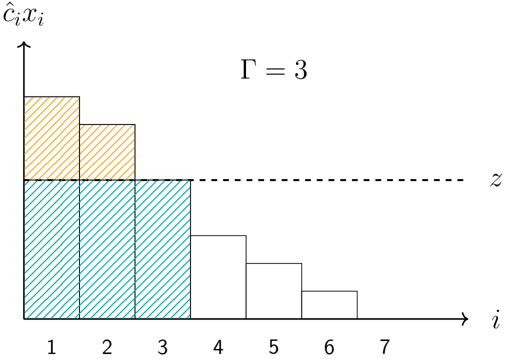

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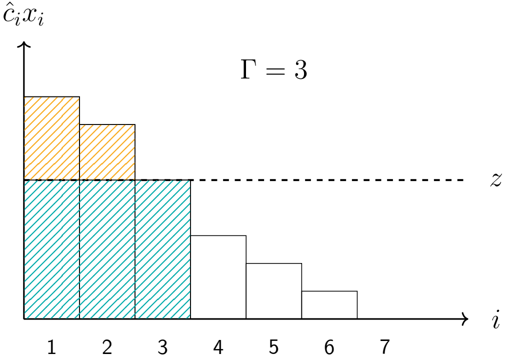

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- Prune z on the fly using relations between objective values [HRS2018]

$$\min \quad \Gamma z' + \sum_{i \in N} (c_i + (\hat{c}_i - z')^+) x_i$$

$$\text{s.t.} \quad Ax \geq b$$

$$x \in \{0,1\}^n$$

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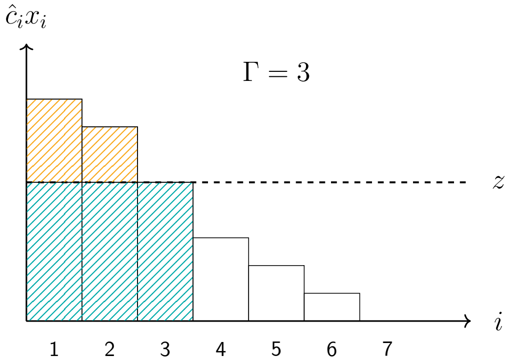

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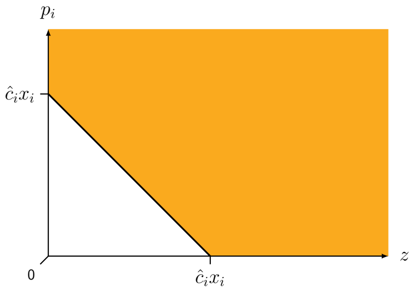

- original constraint $p_i \geq \hat{c}_i x_i - z$

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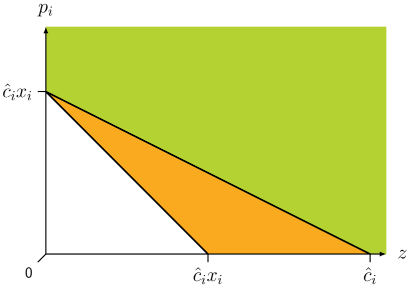

- original constraint $p_i \geq \hat{c}_i x_i - z$
- bilinear constraint $p_i \geq \hat{c}_i x_i - x_i z$

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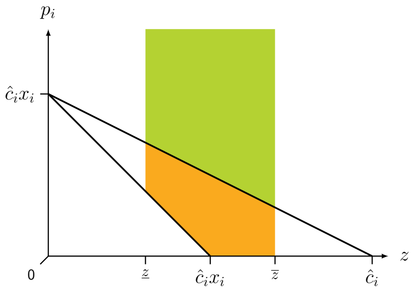

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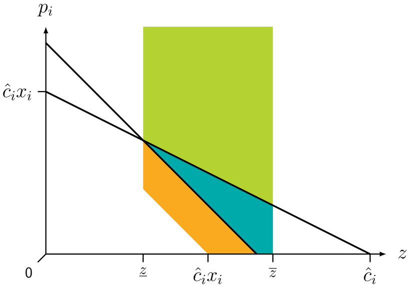

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$$p_i \geq (\hat{c}_i - \underline{z})x_i + \underline{z} - z \qquad (1)$$

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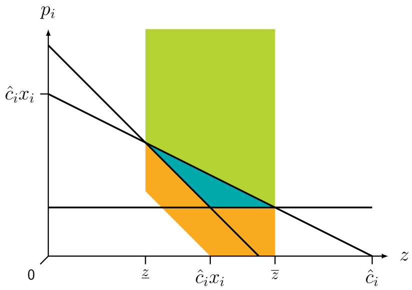

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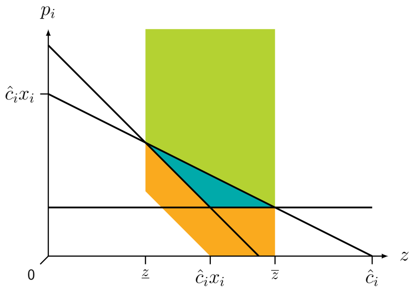
$$p_i \geq (\hat{c}_i - \underline{z})x_i + \underline{z} - z \qquad (1)$$

and

$$p_i \geq (\hat{c}_i - \overline{z})x_i \qquad (2)$$

- an optimal $z \in \{0, \hat{c}_1, \ldots, \hat{c}_n\}$ exists
- fixing z yields nominal problem $\Rightarrow$ solve $n+1$ nominal problems [BS2003]
- $n+1$ may be too large for brute enumeration $\Rightarrow$ bound z instead of fixing it

- consider feasible values of p_i, z for fixed $x_i \in (0,1)$



- original constraint $p_i \geq \hat{c}_i x_i - z$
- bilinear constraint $p_i \geq \hat{c}_i x_i - x_i z$
- assume we are given bounds $\underline{z} \leq z \leq \overline{z}$
- we linearize the bilinear constraint to

$$p_i \geq (\hat{c}_i - \underline{z})x_i + \underline{z} - z \qquad (1)$$

and

$$p_i \geq (\hat{c}_i - \overline{z})x_i \qquad (2)$$

Proposition

Inequalities (1) and (2) approximate the bilinear one and are equally strong if $z \in \{\underline{z}, \overline{z}\}$.

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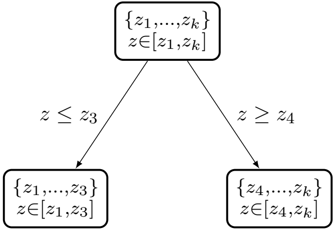
- branch and restrict z to new domains



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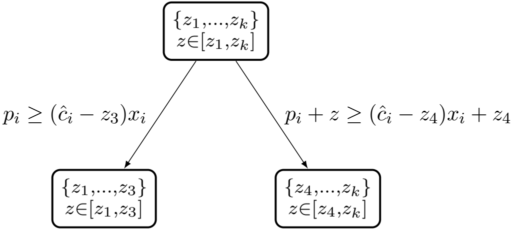

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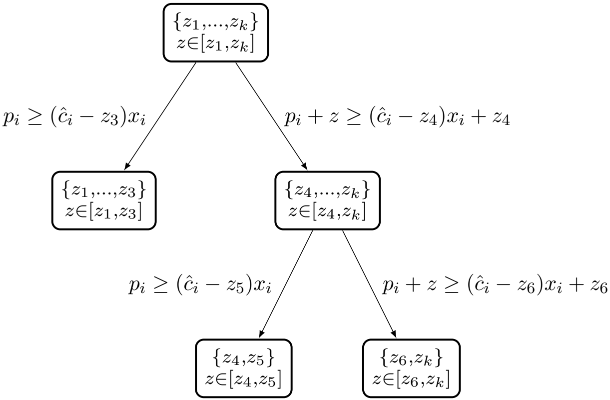

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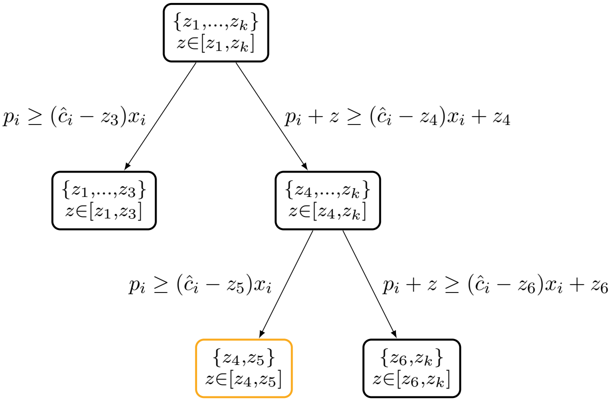

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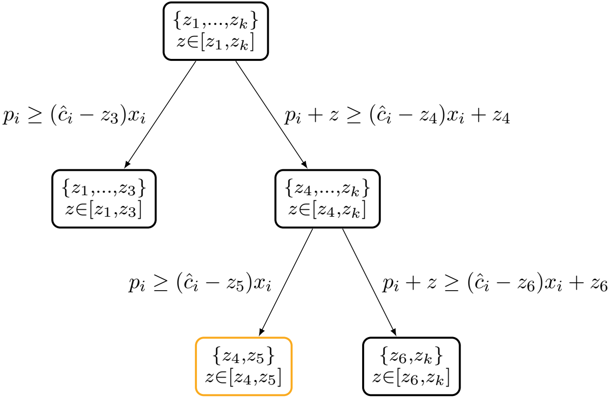

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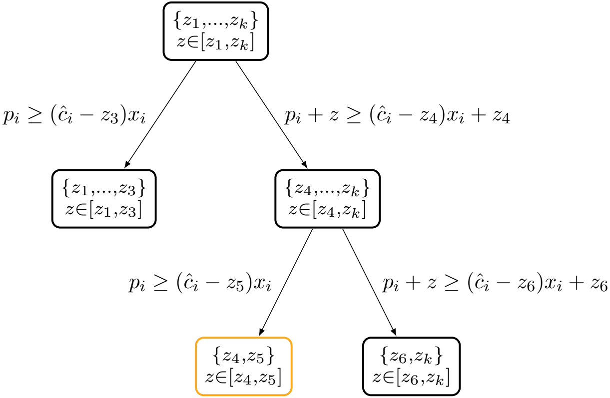

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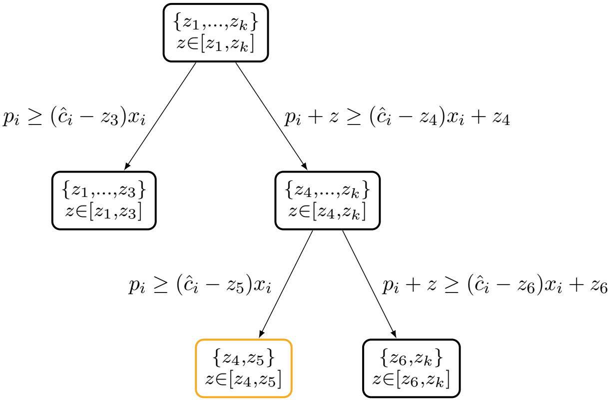

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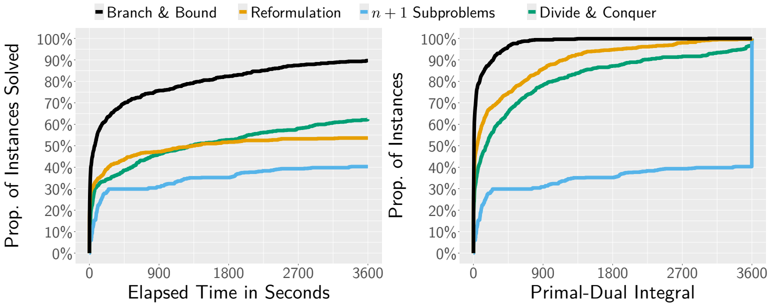
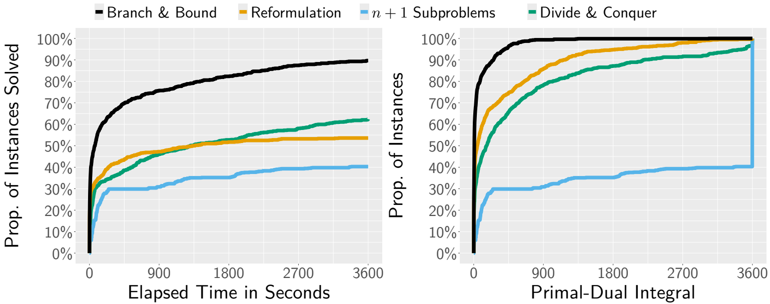
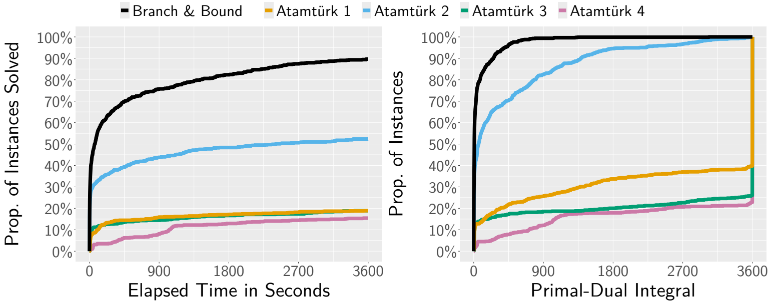
  - Bertsimas Sim standard reformulation, $|\mathcal{Z}|$ nominal subproblems, Divide & Conquer [HRS2018], Atamtürk’s formulations,



Branch & Bound
Reformulation
$n+1$ Subproblems
Divide & Conquer
Prop. of Instances Solved
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
0
900
1800
2700
3600
Elapsed Time in Seconds
Prop. of Instances
Primal-Dual Integral

Computational Results: B&B vs. Literature





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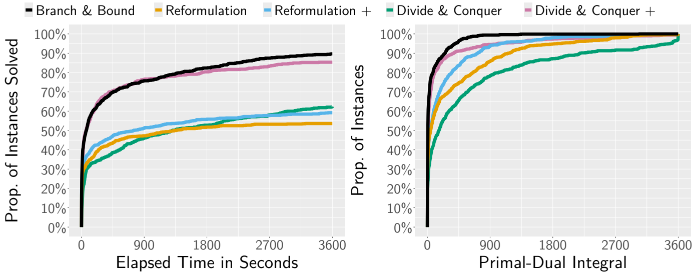

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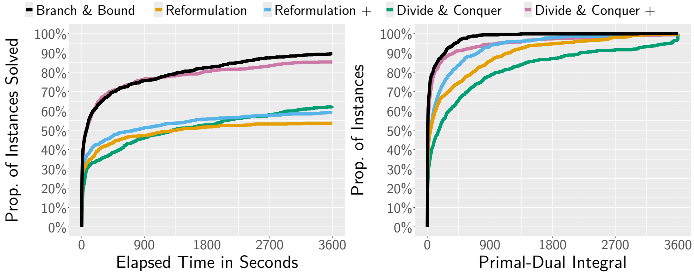

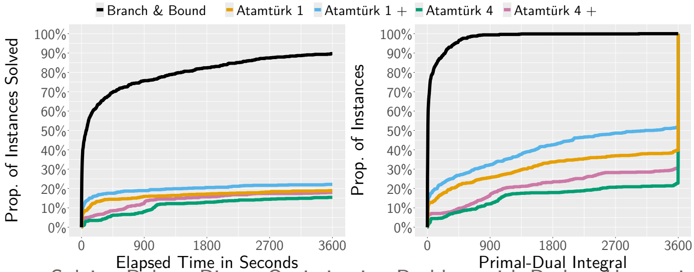

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Theorem

Let $\sum_{i\in N} \pi_i x_i \leq \pi_0$ be a recyclable inequality. Then the ***recycled inequality***

$$\sum_{i\in N} \pi_i p_i + z\pi_0 \geq \sum_{i\in N} \pi_i \hat{c}_i x_i$$

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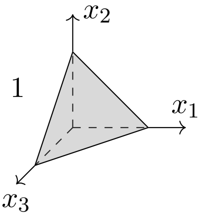

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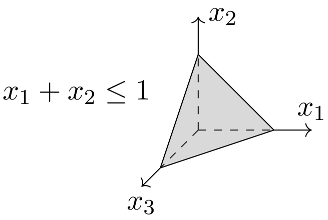

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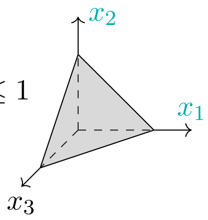

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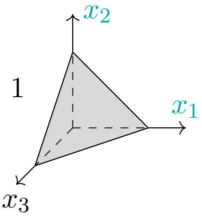

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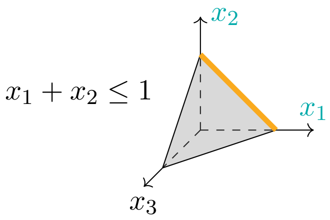

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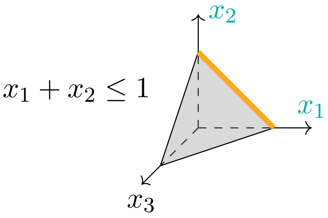

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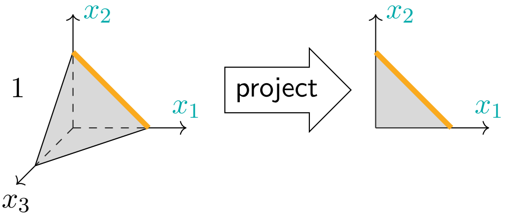

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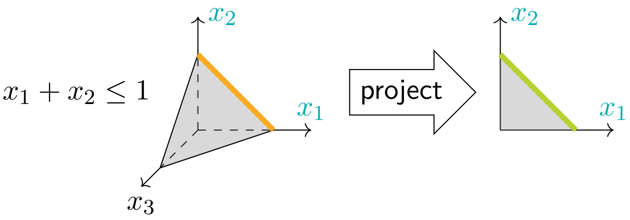

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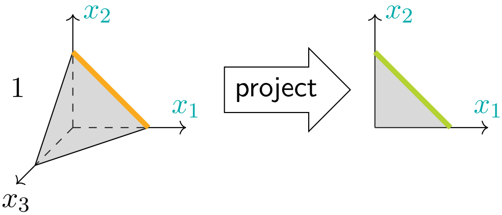

$p_1 + p_2 + z \geq \hat{c}_1 x_1 + \hat{c}_2 x_2$
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Corollary

Assume that $\mathcal{C}^{\mathrm{NOM}}$ *is full-dimensional. If* $\sum_{i \in N} \pi_i x_i \leq \pi_0$ *is recyclable and facet-defining for* $\mathcal{C}^{\mathrm{NOM}}$*, then its recycled inequality is facet-defining for* $\mathcal{C}^{\mathrm{ROB}}$*.*

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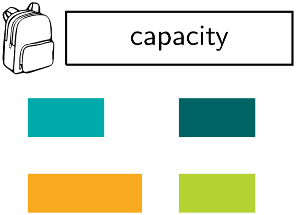

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- let $\sum_{i \in C} x_i \leq |C| - 1$ be a minimal cover inequality

Facet-Defining Recycled Inequalities

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Dominated inequalities can also yield facet-defining recycled inequalities.

Robust Knapsack

- let $\sum_{i \in C} x_i \leq |C| - 1$ be a minimal cover inequality
- in general not facet-defining for knapsack

Facet-Defining Recycled Inequalities

Corollary

Assume that $\mathcal{C}^{\text{NOM}}$ is full-dimensional. If $\sum_{i\in N}\pi_i x_i \leq \pi_0$ is recyclable and facet-defining for $\mathcal{C}^{\text{NOM}}$, then its recycled inequality is facet-defining for $\mathcal{C}^{\text{ROB}}$.

- corollary can be generalized to problems with lower dimension

Observation

Dominated inequalities can also yield facet-defining recycled inequalities.

Robust Knapsack

- let $\sum_{i\in C} x_i \leq |C|-1$ be a minimal cover inequality
- in general not facet-defining for knapsack
- but $\sum_{i\in C} p_i + (|C|-1)z \geq \sum_{i\in C} \hat{c}_i x_i$ is always facet-defining for robust knapsack

Computational Study: Bipartite Matching

- standard formulation + recycle constraints $\sum_{e\in\delta(v)} x_e \leq 1$ for all nodes v

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nodes	timeout	time	P-D integral	int. gap	timeout	time	P-D integral	int. gap
50	0	1.73	0.04	19.53%	0	0.48	0.04	0.33%
100	9	2269.14	3.49	22.82%	0	4.50	0.16	0.32%
150	7	2223.68	2.56	23.66%	0	150.40	0.59	0.27%

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- ~ 504-times faster for 100 nodes and ~ 15-times faster for 150 nodes
- ~ 22-times smaller primal-dual integral for 100 nodes and ~ 4-times smaller for 150 nodes

	robust standard formulation				recycle constraints			
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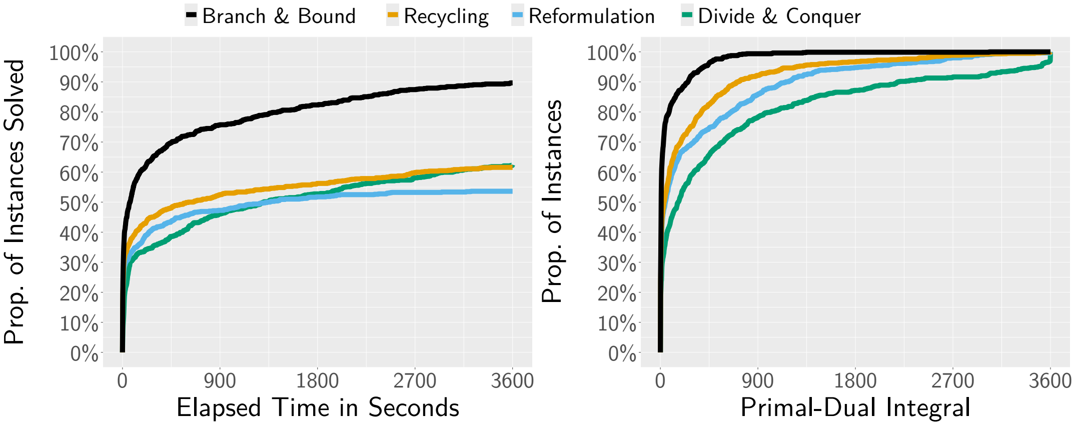
Branch & Bound
Recycling
Reformulation
Divide & Conquer
Prop. of Instances Solved
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
0
900
1800
2700
3600
Elapsed Time in Seconds
Prop. of Instances
Primal-Dual Integral

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- B&B for robust optimization based on strong bilinear formulation
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Thank you for your attention!